

BEST POLYNOMIAL APPROXIMATION ON THE UNIT SPHERE AND THE UNIT BALL

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ABSTRACT. This is a survey on best polynomial approximation on the unit sphere and the unit ball. The central problem is to describe the approximation behavior of a function by polynomials via smoothness of the function. A major effort is to identify a correct gadget that characterizes smoothness of functions, either a modulus of smoothness or a K -functional, the two of which are often equivalent. We will concentrate on characterization of best approximations, given in terms of direct and converse theorems, and report several moduli of smoothness and K -functionals, including recent results that give a fairly satisfactory characterization of best approximation by polynomials for functions in L^p spaces, the space of continuous functions, and Sobolev spaces.

1. INTRODUCTION

One of the central problems in approximation theory is to characterize the error of approximation of a function by the smoothness of the function. In this paper we give a short survey on best approximation by polynomials on the unit sphere \mathbb{S}^{d-1} and the unit ball \mathbb{B}^d in \mathbb{R}^d with

$$\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\} \quad \text{and} \quad \mathbb{B}^d = \{x : \|x\| \leq 1\},$$

where $\|x\|$ denotes the Euclidean norm of x . To get a sense of the main problem and its solution, let us consider first \mathbb{S}^1 and \mathbb{B}^1 .

If we parametrize \mathbb{S}^1 by $(\cos \theta, \sin \theta)$ with $\theta \in [0, 2\pi)$ and identify a function f defined on \mathbb{S}^1 with the 2π periodic function $g(\theta) = f(\cos \theta, \sin \theta)$, then polynomials on \mathbb{S}^1 are precisely trigonometric polynomials, so that polynomial approximation of functions on the circle \mathbb{S}^1 is the same as trigonometric approximation of 2π -periodic functions. Let \mathcal{T}_n denote the space of trigonometric polynomials of degree at most n , $\mathcal{T}_n := \{a_0 + \sum_{k=1}^n a_k \cos k\theta + b_k \sin k\theta : a_k, b_k \in \mathbb{R}\}$. Let $\|\cdot\|_p$ denote the $L^p(\mathbb{S}^1)$ norm of 2π -periodic functions on $[0, 2\pi)$ if $1 \leq p < \infty$, and the uniform norm of $C(\mathbb{S}^1)$ if $p = \infty$. For $f \in L^p(\mathbb{S}^1)$ if $1 \leq p < \infty$, or $f \in C(\mathbb{S}^1)$ if $p = \infty$, define

$$E_n(f)_p := \inf_{t_n \in \mathcal{T}_n} \|f - t_n\|_p,$$

the error of best approximation by trigonometric polynomials. The convergence behavior of $E_n(f)_p$ is usually characterized by a modulus of smoothness. For $f \in L^p(\mathbb{S}^1)$ if $1 \leq p < \infty$ or $f \in C(\mathbb{S}^1)$ if $p = \infty$, $r = 1, 2, \dots$ and $t > 0$, the modulus of smoothness defined by the forward difference is

$$\omega_r(f; t)_p := \sup_{|\theta| \leq t} \left\| \vec{\Delta}_\theta^r f \right\|_p, \quad 1 \leq p \leq \infty,$$

where $\vec{\Delta}_h f(x) := f(x+h) - f(x)$ and $\vec{\Delta}_h^r := \vec{\Delta}_h^{r-1} \vec{\Delta}_h$. The characterization of best approximation on \mathbb{S}^1 is classical (cf. [11, 26]).

Theorem 1.1. For $f \in L^p(\mathbb{S}^1)$ if $1 \leq p < \infty$ or $f \in C(\mathbb{S}^1)$ if $p = \infty$,

$$(1.1) \quad E_n(f)_p \leq c \omega_r(f; n^{-1})_p, \quad 1 \leq p \leq \infty, \quad n = 1, 2, \dots$$

On the other hand,

$$(1.2) \quad \omega_r(f; n^{-1})_p \leq c n^{-r} \sum_{k=1}^n k^{r-1} E_{k-1}(f)_p, \quad 1 \leq p \leq \infty.$$

The theorem contains two parts. The direct inequality (1.1) is called the Jackson estimate, its proof requires constructing a trigonometric polynomial that is close to the best approximation. The weak converse inequality (1.2) is called the Bernstein estimate as its proof relies on the Bernstein inequality. Throughout this paper, we let c, c_1, c_2 denote constants independent of f and n . Their values may differ at different times.

Another important gadget, often easier to use in theoretical studies, is the K -functional defined by

$$K_r(f, t)_p := \inf_{g \in W_p^r} \left\{ \|f - g\|_p + t^r \|g^{(r)}\|_p \right\},$$

where W_p^r denotes the Sobolev space of functions whose derivatives up to r -th order are all in $L^p(\mathbb{S}^1)$. The modulus of smoothness $\omega_r(f, t)_p$ and the K -function $K_r(f, t)_p$ are known to be equivalent: for some constants $c_2 > c_1 > 0$, independent of f and t ,

$$(1.3) \quad c_1 K_r(f, t)_p \leq \omega_r(f, t)_p \leq c_2 K_r(f, t)_p.$$

All characterizations of best approximation, either on the sphere \mathbb{S}^{d-1} or on the ball \mathbb{B}^d , encountered in this paper follow along the same line: we need to define an appropriate modulus of smoothness and use it to establish direct and weak converse inequalities; and we can often define a K -functional that is equivalent to the modulus of smoothness.

Convention: In most cases, our direct and weak converse estimates are of the same form as those in (1.1) and (1.2). In those cases, we shall simply state that the direct and weak converse theorems hold and will not state them explicitly.

We now turn our attention to approximation by polynomials on the interval $\mathbb{B}^1 := [-1, 1]$. Let Π_n denote the space of polynomials of degree n and let $\|\cdot\|_p$ also denote the L^p norm of functions on $[-1, 1]$ as in the case of \mathbb{S}^1 . For $f \in L^p(\mathbb{B}^1)$, $1 \leq p < \infty$, or $f \in C(\mathbb{B}^1)$ for $p = \infty$, define

$$E_n(f)_p := \inf_{t_n \in \Pi_n} \|f - p_n\|_p, \quad 1 \leq p \leq \infty.$$

The difficulty in characterizing $E_n(f)_p$ lies in the difference between approximation behavior at the interior and at the boundary of \mathbb{B}^1 . It is well known that polynomial approximation on \mathbb{B}^1 displays a better convergence behavior at points close to the boundary than at points in the interior. A modulus of smoothness that is strong enough for both direct and converse estimates should catch this boundary behavior.

There are several successful definitions of modulus of smoothness in the literature. The most satisfactory one is due to Ditzian and Totik in [15]. For $r \in \mathbb{N}$ and $h > 0$, let $\widehat{\Delta}_h^r$ denote the central difference of increment h , defined by

$$(1.4) \quad \widehat{\Delta}_h f(x) = f(x + \tfrac{h}{2}) - f(x - \tfrac{h}{2}) \quad \text{and} \quad \widehat{\Delta}_h^r = \widehat{\Delta}_h^{r-1} \Delta, \quad r = 2, 3, \dots$$

Let $\varphi(x) := \sqrt{1-x^2}$. For $r = 1, 2, \dots$, and $1 \leq p \leq \infty$, the Ditzian-Totik moduli of smoothness are defined by

$$(1.5) \quad \omega_\varphi^r(f, t)_p := \sup_{0 < h \leq t} \left\| \widehat{\Delta}_{h\varphi}^r f \right\|_{L^p[-1,1]},$$

where $\widehat{\Delta}_{h\varphi}^r f(x) = 0$ if $x \pm rh\varphi(x)/2 \notin [-1, 1]$. Both direct theorem and weak converse theorem for $E_n(f)_p$ hold for this modulus of smoothness. Furthermore, the K -functional that is equivalent to this modulus of smoothness is defined by, for $t > 0$ and $r = 1, 2, \dots$,

$$(1.6) \quad K_{r,\varphi}(f, t)_p := \inf_{g \in C^r[-1,1]} \left\{ \|f - g\|_p + t^r \|\varphi^r g^{(r)}\|_p \right\}.$$

In the rest of this paper, we discuss characterization of the best approximation on the sphere \mathbb{S}^{d-1} and on the ball \mathbb{B}^d . The problem for higher dimension is much harder. For example, functions on \mathbb{S}^{d-1} are no longer periodic, and there are interactions between variables for functions on \mathbb{S}^{d-1} and \mathbb{B}^d .

The paper is organized as follows. The characterization of best approximation on the sphere is discussed in the next section, and the characterization on the ball is given in Section 3. In Section 4 we discuss recent result on Sobolev approximation on the ball, which are useful for spectral methods for numerical solution of partial differential equations. The paper ends with a problem on characterizing best polynomial approximation of functions in Sobolev spaces.

2. APPROXIMATION ON THE UNIT SPHERE

We start with necessary definitions on polynomial spaces and differential operators.

2.1. Spherical harmonics and spherical polynomials. For \mathbb{S}^{d-1} with $d \geq 3$, spherical harmonics play the role of trigonometric functions for the unit circle. There are many books on spherical harmonic – we follow [10]. Let \mathcal{P}_n^d denote the space of real homogeneous polynomials of degree n and let Π_n^d denote the space of real polynomials of degree at most n . It is known that

$$\dim \mathcal{P}_n^d = \binom{n+d-1}{n} \quad \text{and} \quad \dim \Pi_n^d = \binom{n+d}{n}.$$

Let $\Delta := \partial_1^2 + \dots + \partial_d^2$ denote the usual Laplace operator. A polynomial $P \in \Pi_n^d$ is called harmonic if $\Delta P = 0$. For $n = 0, 1, 2, \dots$ let $\mathcal{H}_n^d := \{P \in \mathcal{P}_n^d : \Delta P = 0\}$ be the linear space of real harmonic polynomials that are homogeneous of degree n . Spherical harmonics are the restrictions of elements in \mathcal{H}_n^d on the unit sphere. It is known that

$$a_n^d := \dim \mathcal{H}_n^d = \dim \mathcal{P}_n^d - \dim \mathcal{P}_{n-2}^d.$$

Let $\Pi_n^d(\mathbb{S}^{d-1})$ denote the space of polynomials restricted on \mathbb{S}^{d-1} . Then

$$\Pi_n^d(\mathbb{S}^{d-1}) = \bigoplus_{0 \leq j \leq n/2} \mathcal{H}_{n-2j}^d \Big|_{\mathbb{S}^{d-1}} \quad \text{and} \quad \dim \Pi_n^d(\mathbb{S}^{d-1}) = \dim \mathcal{P}_n^d + \dim \mathcal{P}_{n-1}^d.$$

For $x \in \mathbb{R}^d$, write $x = r\xi$, $r \geq 0$, $\xi \in \mathbb{S}^{d-1}$. The Laplace operator can be written as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_0,$$

where Δ_0 is a differential operator on ξ , called the Laplace-Beltrami operator; see [10, Section 1.4]. The spherical harmonics are eigenfunctions of Δ_0 . More precisely,

$$\Delta_0 Y(\xi) = -n(n + d - 2)Y(\xi), \quad Y \in \mathcal{H}_n^d.$$

The spherical harmonics are orthogonal polynomials on the sphere. Let $d\sigma$ be the surface measure, and ω_{d-1} be the surface area of \mathbb{S}^{d-1} . For $f, g \in L^1(\mathbb{S}^{d-1})$, define

$$\langle f, g \rangle_{\mathbb{S}^{d-1}} := \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} f(\xi)g(\xi)d\sigma(\xi).$$

If $Y_n \in \mathcal{H}_n^d$ for $n = 0, 1, \dots$, then $\langle Y_n, Y_m \rangle_{\mathbb{S}^{d-1}} = 0$ if $n \neq m$. A basis $\{Y_\nu^n : 1 \leq \nu \leq a_n^d\}$ of \mathcal{H}_n^d is called orthonormal if $\langle Y_\nu, Y_\mu \rangle_{\mathbb{S}^{d-1}} = \delta_{\nu, \mu}$. In terms of an orthonormal basis, the reproducing kernel $Z_{n,d}(\cdot, \cdot)$ of \mathcal{H}_n^d can be written as $Z_{n,d}(x, y) = \sum_{1 \leq \nu \leq a_n^d} Y_\nu(x)Y_\nu(y)$, and the addition formula for the spherical harmonics states that

$$(2.7) \quad Z_{n,d}(x, y) = \frac{n + \lambda}{\lambda} C_n^\lambda(\langle x, y \rangle), \quad \lambda = \frac{d-2}{2},$$

where C_n^λ is the Gegenbauer polynomial of one variable. If $f \in L^2(\mathbb{S}^{d-1})$, then the Fourier orthogonal expansion of f can be written as

$$f = \sum_{n=0}^{\infty} \text{proj}_n f, \quad \text{proj}_n : L^2(\mathbb{S}^{d-1}) \mapsto \mathcal{H}_n^d,$$

where the projection operator proj_n can be written as an integral

$$\text{proj}_n f(x) = \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} f(y) Z_{n,d}(x, y) d\sigma(y).$$

For $f \in L^p(\mathbb{S}^{d-1})$, $1 \leq p < \infty$, or $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$, the error of best approximation by polynomials of degree at most n on \mathbb{S}^{d-1} is defined by

$$E_n(f)_p := \inf_{P \in \Pi_n(\mathbb{S}^{d-1})} \|f - P\|_p, \quad 1 \leq p \leq \infty,$$

where the norm $\|\cdot\|_p$ denote the usual L^p norm on the sphere and $\|\cdot\|_\infty$ denote the uniform norm on the sphere. Our goal is to characterize this quantity in terms of some modulus of smoothness. The direct theorem of such a characterization requires a polynomial that is close to the least polynomial that approximates f . For $p = 2$, the n -th polynomial of best approximation is the partial sum,

$$S_n f = \sum_{k=0}^n \text{proj}_k f,$$

of the Fourier orthogonal expansion, as the standard Hilbert space theory shows. For $p \neq 2$, a polynomial of near best approximation can be given in terms of a cut-off function, which is a C^∞ -function η on $[0, \infty)$ such that $\eta(t) = 1$ for $0 \leq \eta(t) \leq 1$ and $\eta(t) = 0$ for $t \geq 2$. If η is such a function, define

$$(2.8) \quad S_{n,\eta} f(x) := \sum_{k=0}^{\infty} \eta\left(\frac{k}{n}\right) \text{proj}_k f(x).$$

Since η is supported on $[0, 2]$, the summation in $S_{n,\eta} f$ can be terminated at $k = 2n - 1$, so that $S_{n,\eta} f$ is a polynomial of degree at most $2n - 1$.

Theorem 2.1. *Let $f \in L^p(\mathbb{S}^{d-1})$ if $1 \leq p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. Then*

- (1) $S_{n,\eta}f \in \Pi_n(\mathbb{S}^{d-1})$ and $S_{n,\eta}f = f$ for $f \in \Pi_n^d(\mathbb{S}^{d-1})$.
- (2) For $n \in \mathbb{N}$, $\|S_{n,\eta}f\|_p \leq c\|f\|_p$.
- (3) For $n \in \mathbb{N}$, there is a constant $c > 0$, independent of f , such that

$$\|f - S_{n,\eta}f\|_p \leq (1 + c)E_n(f)_p.$$

This near-best approximation was used for approximation on the sphere already in [18] and it has become a standard tool by now. For further information, including a sharp estimate of its kernel function, see [10].

2.2. First Modulus of Smoothness and K -functional. The first modulus of smoothness is defined in terms of spherical means.

Definition 2.2. For $0 \leq \theta \leq \pi$ and $f \in L^1(\mathbb{S}^{d-1})$, define the spherical means

$$T_\theta f(x) := \frac{1}{\omega_{d-1}} \int_{\mathbb{S}_x^\perp} f(x \cos \theta + u \sin \theta) d\sigma(u),$$

where $\mathbb{S}_x^\perp := \{y \in \mathbb{S}^{d-1} : \langle x, y \rangle = 0\}$. For $f \in L^p(\mathbb{S}^{d-1})$, $1 \leq p < \infty$, or $C(\mathbb{S}^{d-1})$, $p = \infty$, and $r > 0$, define

$$(2.9) \quad \omega_r^*(f, t)_p := \sup_{|\theta| \leq t} \|(I - T_\theta)^{r/2} f\|_p,$$

where $(I - T_\theta)^{r/2}$ is defined by its formal infinite series when $r/2$ is not an integer.

The equivalent K -functional of this modulus is defined by

$$(2.10) \quad K_r^*(f, t)_p := \inf_g \left\{ \|f - g\|_p + t^r \|(-\Delta_0)^{r/2} g\|_p \right\},$$

where Δ_0 is the Laplace-Beltrami operator on the sphere and the infimum is taken over all g for which $(-\Delta_0)^{r/2} g \in L^p(\mathbb{S}^{d-1})$.

This modulus of smoothness was first defined and studied in [4, 23].

Theorem 2.3. For $1 \leq p \leq \infty$, the modulus of smoothness $\omega_r^*(f, t)_p$ can be used to establish both direct and weak converse theorems, and it is equivalent to $K_r^*(f, t)_p$.

The direct and the weak converse theorems were established in various stages by several authors (see [4, 17, 21, 23, 27] and [22, 27] for further references), before it was finally established in full generality by Rustamov [25]. A complete proof is given in [27] and a simplified proof can be found in [10].

The spherical means T_θ are multiplier operators of Fourier orthogonal series, i.e.,

$$(2.11) \quad \text{proj}_n T_\theta f = \frac{C_n^\lambda(\cos \theta)}{C_n^\lambda(1)} \text{proj}_n f, \quad \lambda = \frac{d-2}{2}, \quad n = 0, 1, 2, \dots$$

This fact plays an essential role for studying this modulus of smoothness.

It should be mentioned that this multiplier approach can be extended to weighted approximation on the sphere, in which $d\sigma$ is replaced by $h_\kappa^2 d\sigma$, where h_κ is a function invariant under a reflection group. The simplest such weight function is of the form

$$h_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa_i}, \quad \kappa_i \geq 0, \quad x \in \mathbb{S}^{d-1},$$

when the group is \mathbb{Z}_2^d . Such weight functions were first considered by Dunkl associated with Dunkl operators. An extensive theory of harmonic analysis for orthogonal expansions with respect to $h_\kappa^2(x) d\sigma$ has been developed (cf. [8, 16]), in parallel with

the classical theory for spherical harmonic expansions. The weighted best approximation in $L^p(h_\kappa^2; \mathbb{S}^{d-1})$ norm was studied in [29], where analogues of the modulus of smoothness $\omega_r^*(f, t)_p$ and K-functional $K_r^*(f, t)_p$ are defined with $\|\cdot\|_p$ replaced by the norm of $L^p(h_\kappa^2; \mathbb{S}^{d-1})$ for h_κ invariant under a reflection group, and a complete analogue of Theorem 2.3 was established.

The advantages of the moduli of smoothness $\omega_r^*(f, t)_p$ are that they are well-defined for all $r > 0$ and they have a relatively simple structure through multipliers. These moduli, however, are difficult to compute even for simple functions.

2.3. Second Modulus of Smoothness and K-functional. The second modulus of smoothness on the sphere is defined through rotations on the sphere. Let $SO(d)$ denote the group of orthogonal matrix of determinant 1. For $Q \in SO(d)$, let $T(Q)f(x) := f(Q^{-1}x)$. For $t > 0$, define

$$O_t := \left\{ Q \in SO(d) : \max_{x \in \mathbb{S}^{d-1}} d(x, Qx) \leq t \right\},$$

where $d(x, y) := \arccos \langle x, y \rangle$ is the geodesic distance on \mathbb{S}^{d-1} .

Definition 2.4. For $f \in L^p(\mathbb{S}^{d-1})$, $1 \leq p < \infty$, or $C(\mathbb{S}^{d-1})$, $p = \infty$, and $r > 0$, define

$$(2.12) \quad \tilde{\omega}_r(f, t)_p := \sup_{Q \in O_t} \|\Delta_Q^r f\|_p, \quad \text{where } \Delta_Q^r := (I - T_Q)^r.$$

For $r = 1$ and $p = 1$, this modulus of smoothness was introduced and used in [5] and further studied in [19]. For studying best approximation on the sphere, these moduli were introduced and investigated by Ditzian in [12] and he defined them for more general spaces, including $L^p(\mathbb{S}^{d-1})$ for $p > 0$.

Theorem 2.5. The modulus of smoothness $\tilde{\omega}_r(f, t)_p$ can be used to establish both direct and weak converse theorems for $1 \leq p \leq \infty$, and it is equivalent to the K-functional $K_r^*(f, t)_p$ for $1 < p < \infty$, but the equivalence fails if $p = 1$ or $p = \infty$.

The direct and weak converse theorems were established in [13] and [12], respectively. The equivalence of $\tilde{\omega}_r(f; t)_p$ and $K_r^*(f, t)_p$ for $1 < p < \infty$ was proved in [7], and the failure of the equivalence for $p = 1$ and ∞ was shown in [14].

The equivalence passes to the moduli of smoothness and shows, in particular, that $\tilde{\omega}_r(f; t)_p$ is equivalent to the first modulus of smoothness $\omega_r^*(f; t)_p$ for $1 < p < \infty$ but not for $p = 1$ and $p = \infty$.

One advantage of the second moduli of smoothness $\tilde{\omega}_r(f; t)_p$ is that they are independent of the choice of coordinates. These moduli, however, are also difficult to compute even for fairly simple functions.

2.4. Third modulus of smoothness and K-functional. The third modulus of smoothness on the sphere is defined in terms of moduli of smoothness of one variable on multiple circles. For $1 \leq i, j \leq d$, we let $\Delta_{i,j,t}^r$ be the r -rh forward difference acting on the angle of the polar coordinates on the (x_i, x_j) plane. For instance, take $(i, j) = (1, 2)$ as an example,

$$\Delta_{1,2,\theta}^r f(x) = \overrightarrow{\Delta}_\theta^r f(x_1 \cos(\cdot) - x_2 \sin(\cdot), x_1 \sin(\cdot) + x_2 \cos(\cdot), x_3, \dots, x_d).$$

Notice that if $(x_i, x_j) = s_{i,j}(\cos \theta_{i,j}, \sin \theta_{i,j})$ then

$$(x_1 \cos \theta - x_2 \sin \theta, x_1 \sin \theta + x_2 \cos \theta) = s_{i,j} \cos(\theta_{i,j} + \theta),$$

so that $\Delta_{1,2,\theta}^r f(x)$ can be regarded as a difference on the circle of the (x_i, x_j) plane.

Definition 2.6. For $r = 1, 2, \dots$, $t > 0$, and $f \in L^p(\mathbb{S}^{d-1})$, $1 \leq p < \infty$, or $f \in C(\mathbb{S}^{d-1})$ for $p = \infty$, define

$$(2.13) \quad \omega_r(f, t)_p := \max_{1 \leq i < j \leq d} \sup_{|\theta| \leq t} \|\Delta_{i,j,\theta}^r f\|_p.$$

The equivalent K -functional is defined using the angular derivative

$$D_{i,j} := x_i \partial_j - x_j \partial_i = \frac{\partial}{\partial \theta_{i,j}}, \quad 1 \leq i \neq j \leq d$$

where $\theta_{i,j}$ is the angle of polar coordinates in (x_i, x_j) -plane defined as above. For $r \in \mathbb{N}_0$ and $t > 0$, the K -functional is defined by

$$(2.14) \quad K_r(f, t)_p := \inf_g \left\{ \|f - g\|_p + t^r \max_{1 \leq i < j \leq d} \|D_{i,j}^r g\|_p \right\},$$

where g is taken over all $g \in L^p(\mathbb{S}^{d-1})$ for which $D_{i,j}^r g \in L^p(\mathbb{S}^{d-1})$ for all $1 \leq i, j \leq d$.

Theorem 2.7. The modulus of smoothness $\omega_r(f, t)_p$ can be used to establish both direct and weak converse theorems, and is equivalent to $K_r(f, t)_p$ for $1 \leq p \leq \infty$.

These moduli and K -functionals were introduced in [8], where the above theorem was proved. Furthermore, it was also shown that

$$K_r(f, n^{-1})_p \sim \|f - S_{n,\eta} f\|_p + n^{-r} \max_{1 \leq i < j \leq d} \|D_{i,j}^r S_{n,\eta} f\|_p,$$

where $S_{n,\eta}$ is the polynomial defined in (2.8).

For comparison with the other two moduli of smoothness, it was proved in [8] that for $r = 1, 2, \dots$ and $1 \leq p \leq \infty$,

$$\omega_r(f, t)_p \leq \tilde{\omega}_r(f, t)_p, \quad 0 < t < 1.$$

Furthermore, for $1 < p < \infty$, the two moduli of smoothness are equivalent if $r = 1$ or $r = 2$. Thus, the direct theorem with $\omega_r(f, t)_p$ is at least not weaker than the one with either one of the other two moduli of smoothness. Furthermore, all three moduli are equivalent if $1 < p < \infty$ and $r = 1$ or 2 . It remains an open problem if $\omega_r(f, t)_p$ is equivalent to other two moduli of smoothness for $1 < p < \infty$ and $r \geq 3$ or for $p = 1$ and $p = \infty$.

The angular derivatives are related to the Laplace-Beltrami operator by

$$\Delta_0 = \sum_{1 \leq i < j \leq d} D_{i,j}^2.$$

Since the K -functional $K_r^*(f, t)_p$ is defined in terms of Δ_0 and the K -function $K_r(f, t)$ is defined in terms of $D_{i,j}$, it indicates that $K_r(f, t)_p$ may be stronger than $K_r^*(f, t)_p$ if we believe that the parts encode more information than the whole.

The main advantage of the modulus of smoothness $\omega_r(f, t)_p$ lies in the fact that it is defined in terms of moduli of smoothness of one variable, which allows us to tap into the well established theory of trigonometric approximation of one variable, and it also means that $\omega_r(f, t)_p$ can be computed relatively easily (see [8] for examples).

One interesting phenomenon observed from the computational example is that the best approximation on \mathbb{S}^{d-1} for $d \geq 3$ displays a boundary behavior rather like approximation by polynomials on $[-1, 1]$. This is not all that surprising on second thought, but it does put $d = 2$ in approximation on \mathbb{S}^{d-1} apart from $d \geq 3$.

3. APPROXIMATION ON THE UNIT BALL

On the unit ball, we often work with weighted approximation with a fairly general weight function. We shall restrict our discussion to the classical weight function

$$w_\mu(x) := (1 - \|x\|^2)^{\mu-1/2}, \quad \mu > -1/2, \quad x \in \mathbb{B}^d,$$

for which the most has been done. We start with an account of orthogonal structure.

3.1. Orthogonal structure on the unit ball. For the weight function W_μ , we consider the space $L^p(w_\mu, \mathbb{B}^d)$ for $1 \leq p < \infty$ or $C(\mathbb{B}^d)$ when $p = \infty$. The norm of the space $L^p(w_\mu, \mathbb{B}^d)$ will be denoted by $\|f\|_{\mu,p}$, taken with the measure $w_\mu(x)dx$. The inner product of $L^2(w_\mu, \mathbb{B}^d)$ is defined by

$$\langle f, g \rangle_{\mu,p} := b_\mu \int_{\mathbb{B}^d} f(x)g(x)w_\mu(x)dx,$$

where b_μ is the normalization constant of w_μ such that $\langle 1, 1 \rangle_{\mu,p} = 1$. Let $\mathcal{V}_n^d(w_\mu)$ denote the space of polynomials of degree n that are orthogonal to polynomials in Π_{n-1}^d with respect to the inner product $\langle \cdot, \cdot \rangle_{\mu,p}$. It is known that $\dim \mathcal{V}_n^d(w_\mu) = \binom{n+d-1}{n}$. The orthogonal polynomials in $\mathcal{V}_n^d(w_\mu)$ are eigenfunctions of a second order differential operator: for $g \in \mathcal{V}_n^d(w_\mu)$,

$$(3.15) \quad \mathcal{D}_\mu g := (\Delta - \langle x, \nabla \rangle^2 - (2\mu + d - 1)\langle x, \nabla \rangle)g = -n(n + 2\mu + d - 1)g.$$

For $\nu \in \mathbb{N}_0^d$ with $|\nu| = n$, let P_ν^n denote an orthogonal polynomial in $\mathcal{V}_n^d(w_\mu)$. If $\{P_\nu^n : |\nu| = n\}$ is an orthonormal basis of \mathcal{V}_n^d , then the reproducing kernel $P_n(w_\mu; \cdot, \cdot)$ of $\mathcal{V}_n^d(w_\mu)$ can be written as $P_n(w_\mu; x, y) = \sum_{|\nu|=n} P_\nu^n(x)P_\nu^n(y)$. This kernel satisfies a closed-form formula ([28]) that will be given later in this subsection. Let $L^2(w_\mu, \mathbb{B}^d)$, then the Fourier orthogonal expansion of f can be written as

$$f = \sum_{n=0}^{\infty} \text{proj}_n^\mu f, \quad \text{proj}_n^\mu : L^2(w_\mu, \mathbb{B}^d) \mapsto \mathcal{V}_n^d(w_\mu),$$

where the projection operator proj_n can be written as an integral

$$\text{proj}_n^\mu f(x) = b_\mu \int_{\mathbb{B}^d} f(y)P_n(w_\mu; x, y)w_\mu(y)dy.$$

For $f \in L^p(w_\mu, \mathbb{B}^d)$, $1 \leq p < \infty$, or $f \in C(\mathbb{B}^d)$ if $p = \infty$, the error of best approximation by polynomials of degree at most n is defined by

$$E_n(f)_{\mu,p} := \inf_{P \in \Pi_n^d} \|f - P\|_{\mu,p}, \quad 1 \leq p \leq \infty.$$

The direct theorem for $E_n(f)_{\mu,p}$ is also established with the help of a polynomial that is a near best approximation to f . For $p = 2$, the best polynomial of degree n is again the partial sum, $S_n^\mu f := \sum_{k=0}^n \text{proj}_k^\mu f$, of the Fourier orthogonal expansion, whereas for $p \neq 2$ we can choose the polynomial as

$$(3.16) \quad S_{n,\eta}^\mu f(x) := \sum_{k=0}^{\infty} \eta\left(\frac{k}{n}\right) \text{proj}_k^\mu f(x),$$

where η is a cut-off function as in (2.8). The analogue of Theorem 2.1 holds for $S_{n,\eta}^\mu$ and $\|\cdot\|_{\mu,p}$ norm.

If μ is an integer or a half integer, then the orthogonal structure of $L^2(w_\mu, \mathbb{B}^d)$ is closely related to the orthogonal structure on the unit sphere, which allows us to

deduce many properties for analysis on the unit ball from the corresponding results on the unit sphere. The connection is based on the following identity: if d and m are positive integers, then for any $f \in L(\mathbb{S}^{d+m-1})$,

$$\int_{\mathbb{S}^{d+m-1}} f(y) d\sigma_{d+m} = \int_{\mathbb{B}^d} (1 - \|x\|^2)^{\frac{m-2}{2}} \left[\int_{\mathbb{S}^{m-1}} f(x, \sqrt{1 - \|x\|^2} \xi) d\sigma_m(\xi) \right] dx.$$

This relation allows us to relate the space $\mathcal{V}_n^d(w_\mu)$ with $\mu = \frac{m-1}{2}$ directly to a subspace of \mathcal{H}_n^{d+m} , which leads to a relation between the reproducing kernels.

For $\mu = \frac{m-1}{2}$, the reproducing kernel $P_n(w_\mu; \cdot, \cdot)$ satisfies, for $m > 1$,

$$P_n(w_\mu; x, y) = \frac{1}{\omega_m} \int_{\mathbb{S}^{m-1}} Z_{n,d+m}((x, x'), (y, \sqrt{1 - \|y\|^2} \xi)) d\sigma_m(\xi),$$

where $(x, x') \in \mathbb{S}^{d+m-1}$ with $x \in \mathbb{B}^d$ and $x' = \|x'\|\xi \in \mathbb{B}^m$ with $\xi \in \mathbb{S}^{m-1}$, and it satisfies, for $m = 1$ and $y_{d+1} = \sqrt{1 - \|y\|^2}$,

$$P_n(w_0; x, y) = \frac{1}{2} [Z_{n,d+m}((x, x'), (y, y_{d+1})) + Z_{n,d+m}((x, x'), (y, -y_{d+1}))].$$

Using the identity (2.7), we can then obtain a closed-form formula for $P_n(w_\mu; \cdot, \cdot)$, which turns out to hold for all real $\mu > -1/2$.

3.2. First Modulus of Smoothness and K -functional. The first modulus of smoothness on the unit ball is an analogue of $\omega_r^*(f, t)_p$ on the sphere, defined in the translation operator T_θ^μ . Let I denote the identity matrix and

$$A(x) := (1 - \|x\|^2)I + x^T x, \quad x = (x_1, \dots, x_d) \in \mathbb{B}^d.$$

For W_μ on \mathbb{B}^d , the generalized translation operator is given by

$$T_\theta^\mu f(x) = b_\mu (1 - \|x\|^2)^{\frac{d-1}{2}} \int_{\Omega} f(\cos \theta x + \sin \theta \sqrt{1 - \|x\|^2} u) (1 - uA(x)u^T)^{\mu-1} du,$$

where Ω is the ellipsoid $\Omega = \{u : uA(x)u^T \leq 1\}$ in \mathbb{R}^d .

Definition 3.1. Let $f \in L^p(W_\mu, \mathbb{B}^d)$ if $1 \leq p < \infty$, and $f \in C(\mathbb{B}^d)$ if $p = \infty$. For $r = 1, 2, \dots$, and $t > 0$, define

$$\omega_r^*(f, t)_{\mu,p} := \sup_{|\theta| \leq t} \|\Delta_{\theta,\mu}^r f\|_{p,\kappa}, \quad \Delta_{\theta,\mu}^r f := (I - T_\theta^\mu)^{r/2} f.$$

The equivalent K -functional is defined via the differential operator \mathcal{D}_μ in (3.15),

$$K_r^*(f, t)_{\mu,p} := \inf_g \{ \|f - g\|_{\mu,p} + t^r \|\mathcal{D}_\mu^r g\|_{\mu,p} \},$$

where g is taken over all $g \in L^p(W_\mu, \mathbb{B}^d)$ for which $\mathcal{D}_\mu^r g \in L_p(W_\mu, \mathbb{B}^d)$.

Theorem 3.2. For $1 \leq p \leq \infty$, the modulus of smoothness $\omega_r^*(f, t)_{\mu,p}$ can be used to establish both direct and weak converse theorems, and it is equivalent to $K_r^*(f, t)_{\mu,p}$.

These moduli of smoothness and K -functionals were defined in [29] and Theorem 3.2 was also proved there. The integral formula of $T_\theta^\mu f$ was found in [30]. In fact, these results were established for more general weight functions of $h_\kappa^2 w_\mu$ with h_κ being a reflection invariant function. The operator T_θ^μ is a multiplier operator and satisfies

$$\text{proj}_n^\mu (T_\theta^\mu f) = \frac{C_n^{\lambda_\mu}(\cos \theta)}{C_n^{\lambda_\mu}(1)} \text{proj}_n^\mu f, \quad \lambda_\mu = \mu + \frac{d-1}{2}, \quad n = 0, 1, \dots,$$

which is an analogue of (2.11). The proof of Theorem 3.2 can be carried out following the proof of Theorem 2.3.

The advantage of the moduli of smoothness $\omega_r^*(f, t)$ are that they are well-defined for all $r > 0$ and their connection to multipliers, just like the first moduli of smoothness on the sphere. These moduli, however, are difficult to compute even for simple functions.

3.3. Second Modulus of Smoothness and K -functional. The second modulus of smoothness is inherited from the third moduli of smoothness on the sphere. With a slight abuse of notation, we write $w_\mu(x) := (1 - \|x\|^2)^{\mu - \frac{1}{2}}$ for either the weight function on \mathbb{B}^d or that on \mathbb{B}^{d+1} , and write $\Delta_{i,j,\theta}^r$ for either the difference operator on \mathbb{R}^d or that on \mathbb{R}^{d+1} . This should not cause any confusion from the context. We denote by \tilde{f} the extension of f defined by

$$\tilde{f}(x, x_{d+1}) = f(x), \quad (x, x_{d+1}) \in \mathbb{B}^{d+1}, \quad x \in \mathbb{B}^d.$$

Definition 3.3. Let $\mu = \frac{m-1}{2}$, $f \in L^p(w_\mu, \mathbb{B}^d)$ if $1 \leq p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p = \infty$. For $r = 1, 2, \dots$, and $t > 0$, define

$$\omega_r(f, t)_{p,\mu} := \sup_{|\theta| \leq t} \left\{ \max_{1 \leq i < j \leq d} \|\Delta_{i,j,\theta}^r f\|_{L^p(\mathbb{B}^d, W_\mu)}, \max_{1 \leq i \leq d} \|\Delta_{i,d+1,\theta}^r \tilde{f}\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})} \right\},$$

where for $m = 1$, $\|\Delta_{i,d+1,\theta}^r \tilde{f}\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})}$ is replaced by $\|\Delta_{i,d+1,\theta}^r \tilde{f}\|_{L^p(\mathbb{S}^d)}$.

The equivalent K -functional is defined in terms of the angular derivatives $D_{i,j}$, and is defined for all $\mu \geq 0$ by

$$K_r(f, t)_{p,\mu} := \inf_{g \in C^r(\mathbb{B}^d)} \left\{ \|f - g\|_{L^p(W_\mu; \mathbb{B}^d)} + t^r \max_{1 \leq i < j \leq d} \|D_{i,j}^r g\|_{L^p(W_\mu; \mathbb{B}^d)} \right. \\ \left. + t^r \max_{1 \leq i \leq d} \|D_{i,d+1}^r \tilde{g}\|_{L^p(W_{\mu-1/2}; \mathbb{B}^{d+1})} \right\},$$

where if $\mu = 0$, then $\|D_{i,d+1}^r \tilde{g}\|_{L^p(W_{\mu-1/2}; \mathbb{B}^{d+1})}$ is replaced by $\|D_{i,d+1}^r \tilde{g}\|_{L^p(\mathbb{S}^d)}$.

Theorem 3.4. Let $\mu = \frac{m-1}{2}$. For $1 \leq p \leq \infty$, the modulus of smoothness $\omega_r(f, t)_{\mu,p}$ can be used to establish both direct and weak converse theorems, and it is equivalent to $K_r(f, t)_{\mu,p}$.

The moduli of smoothness $\omega_r(f, t)_{p,\mu}$ and the K -functionals $K_r(f, t)_{p,\mu}$ were introduced in [8] and Theorem 3.4 was proved there. The proof relies heavily on the correspondence between $L^p(w_\mu, \mathbb{B}^d)$ and $L^p(\mathbb{S}^{d+m-1})$. In the definition of $\omega_r(f, t)_{\mu,p}$, the term that involves the difference of \tilde{f} may look strange but it is necessary, since $\Delta_{i,j,\theta}^r$ are differences in the spherical coordinates.

For comparison with the first modulus of smoothness $\omega_r^*(f, t)_{\mu,p}$, we only have that for $1 < p < \infty$, $r = 1, 2, \dots$ and $0 < t < 1$,

$$\omega_r(f, t)_{p,\mu} \leq c \omega_r^*(f, t)_{p,\mu}.$$

In all other cases, equivalences are open problems. Furthermore, the main results are established only for $\mu = \frac{m-1}{2}$, but they should hold for all $\mu \geq 0$ and perhaps even $\mu > -1/2$, which, however, requires a different proof from that of [8].

One interesting corollary is that, for $d = 1$, $\omega_r(f, t)_{\mu,p}$ defines a modulus of smoothness on $\mathbb{B}^1 = [-1, 1]$ that is previously unknown. For $\mu = \frac{m-1}{2}$, this modulus

is given by, for $f \in L^p(w_\mu, [-1, 1])$,

$$\omega_r(f, t)_{p, \mu} := \sup_{|\theta| \leq t} \left(c_\mu \int_{\mathbb{B}^2} |\Delta_\theta^r f(x_1 \cos(\cdot) + x_2 \sin(\cdot))|^p w_{\mu-\frac{1}{2}}(x) dx \right)^{1/p}.$$

One advantage of the moduli of smoothness is that they can be relatively easily computed. Indeed, they can be computed just like the second modulus of smoothness on the sphere; see [8] for several examples.

3.4. Third modulus of smoothness and K -functional. The third modulus of smoothness on the unit ball is similar to $\omega_r(f, t)_{p, \mu}$, but with the term that involves the difference of \tilde{f} replaced by another term that resembles the difference in the Ditzian–Totik modulus of smoothness. To avoid the complication of the weight function, we state this modulus of smoothness only for $\mu = 1/2$ for which $w_\mu(x) = 1$. In this subsection, we write $\|\cdot\|_p := \|\cdot\|_{1/2, p}$.

Let e_i be the i -th coordinate vector of \mathbb{R}^d and let $\hat{\Delta}_{he_i}^r$ be the r -th central difference in the direction of e_i . More precisely,

$$\hat{\Delta}_{he_i} f(x) := f(x + he_i) - f(x - he_i), \quad \hat{\Delta}_{he_i}^{r+1} f(x) = \hat{\Delta}_{he_i} \hat{\Delta}_{he_i}^r f(x).$$

As in the case of $[-1, 1]$, we assume that $\hat{\Delta}_{he_i}^r$ is zero if either of the points $x \pm r\frac{h}{2}e_i$ does not belong to \mathbb{B}^d .

Definition 3.5. Let $f \in L^p(\mathbb{B}^d)$ if $1 \leq p < \infty$ and $f \in C(\mathbb{B}^d)$ if $p = \infty$. For $r = 1, 2, \dots$ and $t > 0$,

$$\omega_\varphi^r(f, t)_p := \sup_{0 < |h| \leq t} \left\{ \max_{1 \leq i < j \leq d} \|\Delta_{i,j,h}^r f\|_p, \max_{1 \leq i \leq d} \|\hat{\Delta}_{he_i}^r f\|_p \right\}.$$

With $\varphi(x) := \sqrt{1 - \|x\|^2}$, the equivalent K -functional is defined by

$$K_{r, \varphi}(f, t)_p := \inf_{g \in W_p^r(\mathbb{B}^d)} \left\{ \|f - g\|_p + t^r \max_{1 \leq i < j \leq d} \|D_{i,j}^r g\|_p + t^r \max_{1 \leq i \leq d} \|\varphi^r \partial_i^r g\|_p \right\}.$$

Theorem 3.6. For $1 \leq p \leq \infty$, the modulus of smoothness $\omega_\varphi^r(f, t)_{\mu, p}$ can be used to establish both direct and weak converse theorems, where the direct estimate takes the form

$$E_n(f)_p \leq c \omega_\varphi^r(f, n^{-1})_p + n^{-r} \|f\|_p$$

in which the additional term $n^{-r} \|f\|_p$ can be dropped when $r = 1$, and it is equivalent to $K_{r, \varphi}(f, t)$ in the sense that

$$c^{-1} \omega_\varphi^r(f, t)_p \leq K_{r, \varphi}(f, t)_p \leq c \omega_\varphi^r(f, t)_p + c t^r \|f\|_p,$$

where the term $t^r \|f\|_p$ on the right side can be dropped when $r = 1$.

These moduli of smoothness and K -functionals were also defined in [8], and Theorem 3.6 was proved there. For $d = 1$, they agree with the Ditzian–Totik moduli of smoothness and K -functionals. The K -functional $K_{r, \varphi}(f, t)_{\mu, p}$ can be defined by replacing $\|\cdot\|_p$ with $\|\cdot\|_{\mu, p}$ in the definition of $K_{r, \varphi}(f, t)_p$, which were used to prove direct and weak converse theorems for $E_n(f)_{\mu, p}$ in terms of the K -functionals in [8].

For comparison with the second K -functional $K_r(f, t)_{\mu, p}$, which is only defined for $\mu = \frac{m-1}{2}$, $m = 1, 2, \dots$, we know that for $1 \leq p \leq \infty$,

$$K_{1, \varphi}(f, t)_{\mu, p} \sim K_1(f, t)_{\mu, p}$$

and, for $r > 1$, there is a $t_r > 0$ such that

$$K_r(f, t)_{\mu, p} \leq c K_{r, \varphi}(f, t)_{\mu, p} + c t^r \|f\|_{\mu, p}, \quad 0 < t < t_r,$$

where we need to assume that r is odd if $p = \text{infy}$. We can also state the result for comparison of the moduli of smoothness $\omega_{r, \varphi}(f, t)_p$ and $\omega_r(f, t)_{1/2, p}$ accordingly. The other direction of the equivalence for $r = 2, 3, \dots$ remain open.

The advantages of the modulus of smoothness $\omega_{\varphi}^r(f, t)_p$ and the K -functional $\omega_{\varphi}^r(f, t)_p$ are that they are more intuitive, as direct extensions of the Ditzian–Totik modulus of smoothness and K -functional, and that the modulus of smoothness is relatively easy to compute.

4. APPROXIMATION IN THE SOBOLEV SPACE ON THE UNIT BALL

For $r = 1, 2, \dots$ we consider the Sobolev space $W_r^p(\mathbb{B}^d)$ with the norm defined by

$$\|f\|_{W_r^p(\mathbb{B}^d)} = \left(\sum_{|\alpha| \leq r} \|\partial^\alpha f\|_p \right)^{1/p}.$$

The direct theorem given in terms of the K -functional yields immediately an estimate of $E_n(f)_p$ for functions in the Sobolev space. In the spectral method for solving partial differential equations, we often want estimates for the errors of derivative approximation as well. In this section, we again let $\|\cdot\|_p = \|\cdot\|_{1/2, p}$.

Approximation in Sobolev space requires estimates of derivatives. One such result was proved in [9], which includes the following estimates

$$\|D_{i,j}^r(f - S_n^\mu f)\|_{p, \mu} \leq c E_n(D_{i,j}^r f)_{p, \mu}, \quad 1 \leq i < j \leq d,$$

and a similar estimate that involves $D_{i,d+1} \tilde{f}$. However, what we need is an estimate that involves only derivatives ∂^α instead of $D_{i,j}^r$. In this regard, the following result can be established.

Proposition 4.1. *If $f \in W_p^s(\mathbb{B}^d)$ for $1 \leq p < \infty$, or $f \in C^s(\mathbb{B}^d)$ for $p = \infty$, then for $|\alpha| = s$,*

$$(4.17) \quad \|\phi^{|\alpha|/p}(\partial^\alpha f - \partial^\alpha S_{n,\eta} f)\|_p \leq c E_{n-|\alpha|}(\partial^\alpha f)_p \leq c n^{-s} \|f\|_{W_p^s(\mathbb{B}^d)},$$

where $S_{n,\eta} f = S_{n,\eta}^{1/2} f$ is the near-best approximation defined in (3.16).

The estimate (4.17) in the proposition, however, is still weaker than what is needed in the spectral method, which requires an estimate similar to (4.17) but without the term $[\phi(x)]^{|\alpha|/p} = (1 - \|x\|^2)^{|\alpha|/p}$. It turns out that the near-best approximation $S_{n,\eta}$ is inadequate for obtaining such an estimate. What we need is the orthogonal structure of the Sobolev space $W_2^r(\mathbb{B}^d)$.

The orthogonal structure of $W_2^r(\mathbb{B}^d)$ was studied first in [32] for the case $r = 1$, and in [24, 31] for the case $r = 2$, and in [20] for general r . The inner product of $W_2^r(\mathbb{B}^d)$ is defined by

$$\langle f, g \rangle_{-s} := \langle \nabla^s f, \nabla^s g \rangle_{\mathbb{B}^d} + \sum_{k=0}^{\lceil \frac{s}{2} \rceil - 1} \langle \Delta^k f, \Delta^k g \rangle_{\mathbb{S}^{d-1}}.$$

Let $\mathcal{V}_n^d(w_{-s})$ denote the space of polynomials of degree n that are orthogonal to polynomials in Π_{n-1}^d with respect to the inner product $\langle \cdot, \cdot \rangle_{-s}$. Then $\mathcal{V}_n^d(w_{-1})$

satisfies a decomposition

$$\mathcal{V}_n^d(w_{-1}) = (1 - \|x\|^2)\mathcal{V}_{n-2}^d(w_1) \oplus \mathcal{H}_n^d,$$

where \mathcal{H}_n^d is the space of spherical harmonics of degree n , and $\mathcal{V}_n^d(w_{-2})$ satisfies a decomposition

$$\mathcal{V}_n^d(w_{-2}) = (1 - \|x\|^2)^2\mathcal{V}_{n-4}^d(w_2) \oplus (1 - \|x\|^2)\mathcal{H}_{n-2}^d \oplus \mathcal{H}_n^d.$$

For each of these two cases, an orthonormal basis can be given in terms of the Jacobi polynomials and spherical harmonics, and the basis resembles the basis of $\mathcal{V}_n^d(w_\mu)$ for $\mu = -1$ and $\mu = -2$, which is why we adopt the notation $\mathcal{V}_n^d(w_{-s})$. The pattern of orthogonal decomposition, however, breaks down for $r > 2$. Nevertheless, an orthonormal basis can still be defined for $\mathcal{V}_n^d(w_{-s})$, which allows us to define an analogue of the near-best polynomial $S_{n,\eta}^{-s}f$. The result for approximation in the Sobolev space is as follows.

Theorem 4.2. *Let $r, s = 1, 2, \dots$ and $r \geq s$. If $f \in W_p^r(\mathbb{B}^d)$ with $r \geq s$ and $1 < p < \infty$. Then, for $n \geq s$,*

$$\|f - S_{n,\eta}^{-s}f\|_{W_p^k(\mathbb{B}^d)} \leq cn^{-r+k}\|f\|_{W_p^r(\mathbb{B}^d)}, \quad k = 0, 1, \dots, s,$$

where $S_{n,\eta}^{-s}f$ can be replaced by $S_n^{-s}f$ if $p = 2$.

This theorem is established in [20], which contains further refinements of such estimates in Sobolev spaces. The proof of this theorem, however, requires substantial work and uses a duality argument that requires $1 < p < \infty$.

The estimate in the theorem can be used to obtain an error estimate for the Galerkin spectral method, which looks for approximate solutions of a partial differential equations that are polynomials written in terms of orthogonal polynomials on the ball and their coefficients are determined by the Galerkin method. We refer to [20] for applications on a Helmholtz equation of second order and a biharmonic equation of fourth order on the unit ball. The method can also be applied to Poisson equations consider in [1, 2, 3].

These results raise the question of characterizing the best approximation by polynomials in Sobolev spaces, which is closely related to simultaneous approximation traditionally studied in approximation theory. But there are also distinct differences as the above discussion shows. We end this paper by formulating this problem in a more precise form.

Let Ω be a domain in \mathbb{R}^d and w be a weight function on Ω . For $s = 1, 2, \dots$, and $f \in W_p^s(w, \Omega)$. Define

$$E_n(f)_{W_p^s(w, \Omega)} := \inf_{p_n \in \Pi_n^d} \|f - p_n\|_{W_p^s(w, \Omega)}.$$

Problem 4.3. Establish direct and (weak) converse estimates of $E_n(f)_{W_p^s(w,\Omega)}$.

In the case of $\Omega = \mathbb{B}^d$ and $w(x) = 1$, Theorem 4.2 gives a direct estimate of $E_n(f)_{W_p^s(w,\Omega)}$ for $f \in W_p^r(w,\Omega)$ with $r \geq s$. However, the estimate is weaker than what is needed. A direct estimate should imply that $E_n(f)_{W_p^s(w,\Omega)}$ goes to zero as $n \rightarrow \infty$ whenever $f \in W_p^s(w,\Omega)$. What this calls for is an appropriate K -functional, or a modulus of smoothness, for $f \in W_p^s(w,\Omega)$ that characterize the best approximation $E_n(f)_{W_p^s(w,\Omega)}$.

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